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ON 3-SKEIN ISOMORPHISMS OF GRAPHS

R. L. HEMMINGER*, H. A. JUNG and A. K. KELMANS

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It is shown that a 3-skein isomorphism between 3-connected graphs with at least 5 vertices is induced by an isomorphism. These graphs have no loops but may be infinite and have multiple edges.

1. Introduction

A graph S consisting of n openly disjoint paths joining two vertices is called an *n-skein*. Note that S is a 2-skein if and only if S is a circuit. A bijection between the edge sets $E(G)$ and $E(G^*)$ of graphs G and G^* is called an *n-skein isomorphism* if it induces a bijection between the sets of n-skeins contained in G and G^*, respectively.

Whitney proved in [6] that circuit isomorphism of 3-connected finite graphs are induced by isomorphisms. In [7] he observed that one need only assume the 3-connectedness of one of the graphs to obtain the same conclusion. Various generalisations were given in [1], [2], [3], [4] and [5]. The main result of this paper is the following.

Theorem. *Each 3-skein isomorphism from a 3-connected graph with at least 5 vertices onto a graph without isolated vertices is induced by an isomorphism.*

Actually it will be shown that each 3-skein isomorphism from a 3-connected graph with at least 5 vertices is a circuit isomorphism. The theorem then follows by applying Whitney's theorem (in a suitably generalized form). On the other hand it is easily seen that each circuit isomorphism is a 3-skein isomorphism. This fact is a special case of Theorem 1 in [3].

In [1] the Theorem was proved for 4-connected graphs, and in [3] for 3-connected graphs containing at least one 4-skein. These results were special cases of results about n-skeins.

The theorem in the present form was proved by Kelmans and independently by Hemminger and Jung.

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* The paper was written while this author was visiting Universität für Bildungswissenschaften, Klagenfurt, Austria.

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2. Proof of the Theorem

In all that follows let G, G^* and σ be as in the Theorem; i.e. G is 3-connected and has at least 5 vertices, G^* has no isolated vertices and σ is a 3-skein isomorphism from G onto G^*. For a subgraph H of G with $E(H)\neq\emptyset$, let H' denote the unique subgraph of G^* without isolated vertices that has $E(H')=\sigma(E(H))$. A nontrivial path P in H is considered a subgraph of H and is called a *constituent path of* H if the inner vertices of P, and only these, have valency 2 in H.

Observe that for a path P in G the image P' need not to be a path; even if P' is a path, incident edges need not be mapped by σ onto incident edges.

We will achieve the proof in stages. In the first stage we consider homeomorphs (some authors say subdivisions) of K_4, the complete graph on 4 vertices. Note that each permutation of the edges of K_4 is a 3-skein automorphism of K_4, hence the requirement that G have at least 5 vertices.

Lemma 1. *If H is a homeomorph of K_4 in G, then H' is a homeomorph of K_4 in G^*. Moreover, constituent paths of H are mapped onto constituent paths of H'.*

Proof. Let H be a homeomorph of K_4, and let P be a constituent path of H. Further let S denote the unique 3-skein in H such that $E(S)=E(H)-E(P)$. In H' there exists a path Q between distinct vertices of S' such that $E(S')\cap E(Q)=\emptyset$. Then $H'=S'\cup Q$ since $S'\cup Q$ contains at least four 3-skeins while each proper subgraph of H contains at most one 3-skein. Hence $E(Q)=E(P')$ and, by construction, Q is a constituent path of $S'\cup Q$. Since H and consequently H' contains six 3-skeins $S'\cup Q$ must be a homeomorph of K_4. ▮

For a graph H we let $n(H)=|V(H)|$.

Lemma 2. *If P_1, P_2 and P_3 are constituent paths of a 3-skein S in G and $n(P_i)\geqq 3$ for each i, then each P_i' is a constituent path of S'.*

Proof. Since G is 3-connected there exists, for $i=1,2$ and 3, a path J_i, openly disjoint from S, joining an inner vertex of P_i to an inner vertex of some P_j, $j\neq i$; since some pair of J_1, J_2, J_3 is incident with all three of P_1, P_2 and P_3, we can assume that J_1 and J_2 join P_1 and P_2, respectively, to P_3. Now the graph $H_i=S\cup J_i$ is a homeomorph of K_4 and so, by Lemma 1, H_i' is also. For $i=1$ and 2 moreover, H_i' contains P_{3-i}' as a constituent path. Hence P_{3-i}' is a subgraph of some constituent path P_{3-i}^* of S'. But P_i^* $(i=1,2)$ is composed of either one or two constituent paths of H_i', since S' is a 3-skein in H_i'; on the other hand P_i is composed of two constituent paths of H_i, and hence, by Lemma 1, also P_i' is the union of two constituent paths of H_i'. It follows that $P_1'=P_1^*$ and $P_2'=P_2^*$, and hence that P_1', P_2' and P_3' are the constituent paths of S'. ▮

Lemma 3. *If C is a circuit in G with $3\leqq n(C)<n(G)$, then C' is a circuit.*

Proof. Let $y\in V(G)-V(C)$. Since G is 3-connected, there are paths Q_1, Q_2 and Q_3 in G, openly disjoint from each other and C, which join y to distinct vertices on C, say x_1, x_2 and x_3 respectively.

We consider cases and subcases.

(I) $n(C)\geqq 4$. Then there exists a path $P\subset Q_1\cup Q_2\cup Q_3$ such that P joins nonadjacent vertices on C. By Lemma 2, P' is a constituent path of the 3-skein $C'\cup P'$, and hence C' is a circuit.

(II) $n(C)=3$. We consider two possibilities here.

(a) $n(Q_i)\geqq 3$ for some i, say $n(Q_3)\geqq 3$. We further divide this subcase into two.

(a.1) $n(C\cup Q_3)<n(G)$. In this case $n(C\cup Q_3\cup Q_i)<n(G)$ for $i=1$ or 2, say for $i=1$. Now the path $P=Q_1\cup Q_3$ is a constituent path of the 3-skein $S=C\cup P$ and P is a subgraph of two different circuits C_1 and C_2 in S. Moreover $4\leqq n(C_i)<n(G)$. Hence, by Case I, C_1' and C_2' are circuits in the 3-skein S'. Since $C_1'\cap C_2'=P'$, P' is a constituent path of $S'=C'\cup P'$ and so C' is a circuit.

(a.2) $n(C\cup Q_3)=n(G)$. Since G is 3-connected, some inner vertex x_4 of Q_3 is adjacent to x_1 or x_2, say to x_1. Let R_3 be the subpath of Q_3 between the vertices x_3 and x_4. Remembering that Q_1 and Q_2 are edges, let e_1, e_2, e_3 and e_4 denote respectively the edges x_1x_3, x_2x_3, x_1x_2 and x_1x_4. And let S be the 3-skein with constituent paths $Q_1\cup e_1$, $Q_2\cup e_2$ and Q_3. Thus, by Lemma 2, the 3-skein S' has constituent paths $Q_1'\cup e_1'$, $Q_2'\cup e_2'$ and Q_3'. Moreover, by applying Lemma 1 to both $S\cup e_3$ and $S\cup e_4$, we have that $S'\cup e_3'\cup e_4'$ is as in the figure where $\{Q_2', e_2'\}=\{e', f'\}$ and Q_3' is on the left. The only question is whether $e_2'=e'$ or f'.

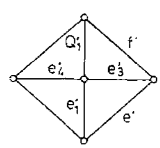

Fig. 1

But $S_1=C\cup e_4\cup R_3$ is a 3-skein in G so we must have $e_2'=e'$; or S_1' will not be a 3-skein. Hence C' is a circuit.

(b) $n(Q_1)=n(Q_2)=n(Q_3)=2$. Since $n(G)\geqq 5$, there is a vertex $z\neq y$, x_1, x_2, x_3. By the previous cases we can assume that z is also adjacent to x_1, x_2 and x_3. Then, by Lemma 2, the 3-skein S consisting of all edges x_iy and x_iz has constituent paths P_1, P_2 and P_3 which are mapped onto constituent paths of S'. Moreover, by Lemma 1, each pair of the inner vertices of the P_i' are joined by the image of one of the edges x_1x_2, x_1x_3 and x_2x_3. Hence C' is a circuit. ∎

Proof of the theorem. Let C be a circuit in G. We wish to show that C' is a circuit. Hence, by Lemma 3, we can assume that either $n(C)=n(G)$ or $n(C)=2$.

If $n(C)=n(G)$, then we can find an edge e joining nonadjacent vertices on C (since G is 3-connected and $n(G)\geqq 5$). But then, by Lemma 3, the two circuits C_1 and C_2 in $C\cup e$ that contain e are mapped onto circuits in the 3-skein $C'\cup e'$ and since they have only e' in common, C' is a circuit.

If $n(C)=2$, say $V(C)=\{y, z\}$, then there exists a path P in G such that $n(P)\geqq 3$ and $C\cup P$ is a 3-skein. But then, by Lemma 3, $P\cup e$ for each edge e of C is a circuit. It follows that C' is a circuit.

We conclude that σ maps circuits in G onto circuits in G^*. To complete the proof that σ is a circuit isomorphism let C^* be a circuit in G^*. Pick $e\in E(G)$ such that $e' \in E(C^*)$ and let C_1 be a circuit in G containing e. By Lemma 3, C_1' is a circuit, which of course also contains e'. Hence, assuming $C_1' \neq C^*$ (for otherwise we are already done), there exists a 3-skein S^* in G^* such that C^* is a circuit in S^*. Now S^* is the image of some 3-skein S in G and the three circuits in S are mapped onto distinct circuits of S^*, one of which must therefore be C^*.

We have shown that σ is a circuit isomorphism. Hence, by Whitney's theorem on circuit isomorphisms and its extension to infinite graphs with multiple edges [3, 4], the claim now follows. ∎

We note that Theorem 2 in [3], for $n=2$, explicitly says that a circuit isomorphism from a 3-connected graph onto a graph without isolated vertices is induced by an isomorphism.

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H. A. Jung

Fachbereich Mathematik
Technische Universität Berlin
Berlin, West Germany

R. L. Hemminger

Department of Mathematics
Vanderbilt University
Nashville, Tennesse 37235 U.S.A.

A. K. Kelmans

Profsoyuznaya Str. 130
K. 3, kv, 33
117321 Moscow
U. S. S. R.